# Approximate Weighted Vertex Cover ①

Given an undirected graph $G=(V,E)$ and a weight $\omega: V \to \mathbb{R}$ function for the vertex set, the weight of a vertex cover $V' \subseteq V$ is $\sum_{v \in V'} \omega(v) = \omega(V')$.

The problem is to find a vertex cover of minimum weight.

We cast the problem as an integer programming.

Let $x(v)$ be a variable for vertex $v$.

minimize $\sum_{v \in V} \omega(v) x(v)$

s.t.

$$x(u) + x(v) \geq 1 \text{ for } \forall (u,v) \in E$$

$$x(v) \in \{0,1\} \text{ for } \forall v \in V.$$

For each edge $(u,v) \in E$ at least one of $x(u)$ or $x(v)$ has to be 1.

We know that integer programming is NP-hard. So, we convert the integer programming to LP by relaxing that $x(v)$ can take any values between 0 and 1.

$$\text{Minimize } \sum_{v \in V} x(v)\, w(v)$$

s.t.

$$x(u) + x(v) \geq 1 \quad \text{for } \forall (u,v) \in E$$

$$x(v) \leq 1 \quad \text{for } \forall v \in V$$

$$x(v) \geq 0 \quad \text{for } \forall v \in V.$$

Any feasible solution to IP is also feasible for LP. Therefore, an optimal solution to LP is a lower bound for the optimal solution to IP and hence the minimum-weight VC problem.

```
Approx-Min-Weight-VC (G, w)
C := ϕ;
Compute x̄, the optimal solution to LP
for each v ∈ V do
    if x̄(v) ≥ 1/2 then
        C := C ∪ {v}
    endif
endfor
return C.
```

**Theorem.** Approx-Min-Weight-VC is a polynomial-time 2-approximation algorithm.

**Proof.** Since LP is polynomial-time solvable, the algorithm is in P.

Let $C^*$ be an optimal solution to the minimum weight vertex cover problem.

Let $z^*$ be the value of an optimal solution to LP.

Since an optimal cover is a feasible solution to LP, we have

1 .... $z^* \leq w(C^*)$

rounding the fractional values of the variables $\bar{x}(v)$, we produce a set C. We show C is a vertex cover.

- $x(u) + x(v) \geq 1$ for any $(u,v) \in E$

$$\Downarrow$$

at least one of $x(u)$ or $x(v) \geq \frac{1}{2}$.

$$\Downarrow$$

at least one of $u$ or $v$ will be in C.

$$\Downarrow$$

all edges $(u,v) \in E$ are covered by C.

- To see approximation consider

$$
\begin{aligned}
z^* &= \sum_{v \in V} w(v)\, \bar{x}(v) \\
&\geq \sum_{v \in V \mid \bar{x}(v) \geq \frac{1}{2}} w(v)\, \bar{x}(v) \\
&\geq \sum_{v \in V \mid \bar{x}(v) \geq \frac{1}{2}} w(v) \cdot \frac{1}{2} \\
&= \frac{1}{2} \sum_{v \in C} w(v) \\
&= \frac{1}{2} w(C).
\end{aligned}
$$

Combining with 1

$\Rightarrow$ $w(C) \leq 2z^* \leq 2w(C^*)$