Probability

S: sample space {finite or countably infinite}.

event: subsets of S.

Probability distribution $Pr\{\}$ on S:

1. $Pr(A) \geq 0$ for any event A
2. $Pr(S) = 1$
3. $Pr(A \cup B) = Pr(A) + Pr(B)$ for two mutually exclusive events

$$Pr\{\bigcup_i A_i\} = \sum_i Pr(A_i)$$

$$Pr(A) = \sum_{s \in A} Pr(s).$$

$Pr(s) = \frac{1}{|S|}$ if uniform probability distribution.

Probability density function:

A (discrete) random variable X is a function $X: S \to \mathbb{R}$.

Define the event $X = x$ to be $\{s \in S : X(s) = x\}$;

$$P_r(X=x) = \sum_{\{s \in S:\, X(s)=x\}} P_r(s)$$

The function $f(x) = P_r\{X=x\}$ is the probability density function of the random variable X.

$$P_r\{X=x\} \geq 0 \quad \text{and} \quad \sum_x P_r\{X=x\} = 1.$$

Expected value of a random variable:

$$E[X] = \sum_x x\, P_r\{X=x\}$$

$$E[X+Y] = E[X] + E[Y].$$

Randomized Approximation algorithm for Max-3-CNF SAT.

A randomized algorithm for a problem has approximation ratio $\rho(n)$, if for any input size n, the expected cost C satisfies

$$\max\left(\frac{C}{C^*}, \frac{C^*}{C}\right) \leq \rho(n).$$

We call then randomized $\rho(n)$-approximation algorithm.

3-CNF : $(x_1 \vee \bar{x}_1 \vee \bar{x}_2) \wedge (x_3 \vee x_2 \vee x_4)$

Literals : occurrence of a variable or its negation.

Clauses : OR of three literals

3-CNF-SAT : Is the formula satisfiable?

3-CNF-SAT is NP-complete.

Max-3-CNF-SAT: Return an assignment of variables that maximizes the number of satisfiable clauses.

Randomized Algorithm. Set each variable to 1 with probability 1/2 and to 0 with probability 1/2.

We show this is an 8/7-approximation algorithm.

- For simplicity assume that no clause contains a both a variable and its negation. (Think about how to remove this constraint).

Theorem Given a Max-3-CNF SAT problem with n variables $x_1, x_2, \ldots, x_n$ and m clauses, the randomized algorithm above is a 8/7-approximation algorithm.

Proof. For $i = 1, \dots, m$ define indicator random variable

$$Y_i = I(\text{clause } i \text{ is satisfied})$$

$$Y_i = \begin{cases} 1 & \text{if at least one literal is 1 in the } i\text{th clause} \\ 0 & \text{o.w.} \end{cases}$$

- With our assumptions, the settings of the three literals are independent
- A clause is not satisfied with prob. $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.
- prob.$\{$clause i is satisfied$\} = 1 - \frac{1}{8} = \frac{7}{8}$.
- $E[Y_i] = \frac{7}{8}$
- Let $Y = Y_1 + Y_2 + \dots + Y_m$.
- $E[Y] = E\left[\sum_{i=1}^{m} Y_i\right] = \sum_{i=1}^{m} E[Y_i]$

$$= \sum_{i=1}^{m} 7/8$$

$$= \frac{7m}{8}.$$

Since m is the upper bound on the # of satisfiable clause, the approximation ratio $m / 7m/8 = 8/7$.