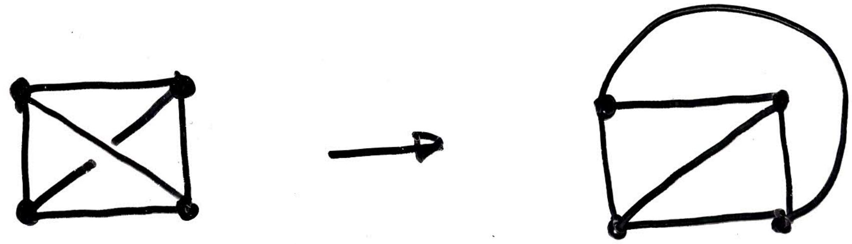
Planar Graphs

A graph that can be drawn in plane without any edge crossings.



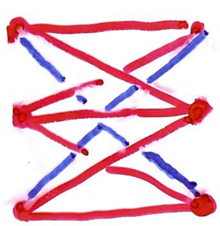
A planar graph drawn with edges crossed.

Same graph drawn without crossing.

Plane embedding: If a drawing of a graph in the plane uses no edge crossings (except at common vertices), then the drawing is a plane embedding.

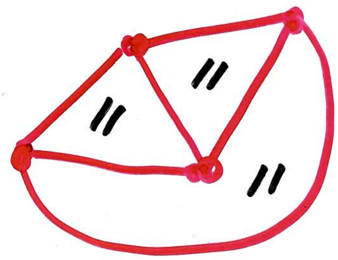
A planar graph has a plane embedding.

A non-planar graph does not have any plane embedding.



$K_{3,3}$ (non-planar)

__Euler's Formula__: When a planar graph is drawn with a plane embedding, there are connected components of $\mathbb{R}^2$ which are called __faces__.



" faces

If V = # vertices, E = # edges, F = # of faces

$$V - E + F = 2$$

__proof__. Choose a spanning tree of (V, E). It's embedding has $F = 1$ face, V vertices and $V-1$ edges.

$$V - (V-1) + 1 = 2$$

So, the formula holds in this case.

Now, add one edge at a time. each time we add an edge, we create a new face.

$$V - (E+1) + (F+1) = V - E + F = 2$$

So, adding an edge does not change the formula.

Vertex coloring

Each vertex is assigned a color so that no two adjacent vertices have the same color.

A k-coloring of (V,E) is a function

$c: V \to \{1, 2, \ldots, k\}$ so that $c(u) \neq c(v)$ if $(u,v) \in E$.

Claim. Every planar graph has at least one vertex of degree at most 5.

Claim Every planar graph is 6-colorable.

proof (V,E) has a vertex of degree less than 6 because $\sum_{v \in V} \deg(v) = 2E \leq 6V - 12$.

Now we give an algorithm.

1. Remove a vertex $v \in V$ with $\deg(v) \leq 5$. Also remove the at most 5 edges incident to v.
2. 6-color the smaller graph recursively.
3. Add v back in and assign a color that is different from the color of its ≤ 5 neighbors.