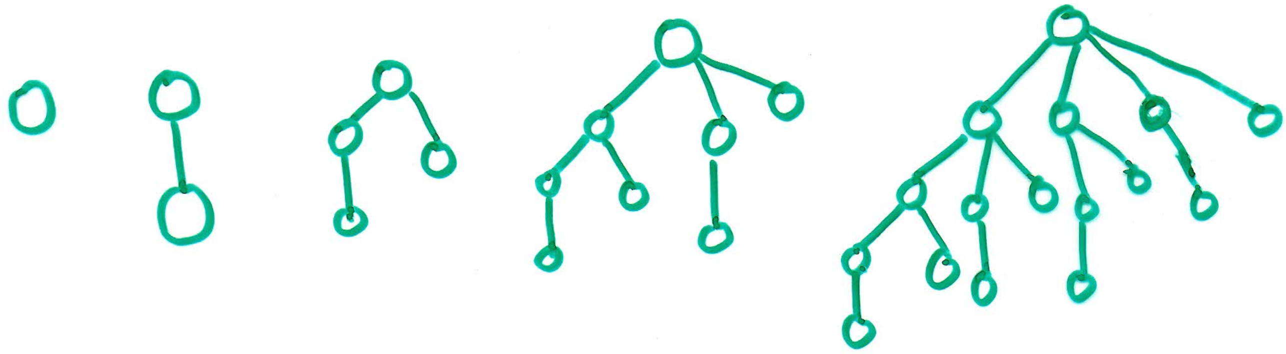
**Binomial Heaps.** The heap-ordered trees in F-heaps are very regular since they are generated either from scratch or by joining two trees of equal degrees at the root.



Binomial heaps

| | | | | | |
|---|---|---|---|---|---|
| number of nodes | 1 | 2 | 4 | 8 | 16 |
| degree of root | 0 | 1 | 2 | 3 | 4 |

**Lemma.** (i) Every tree in an F-heap is a binomial heap

(ii) A tree with the root of degree $k$ has $2^k$ nodes

(iii) Each node in a tree with $n$ nodes has degree $\leq \log_2 n$

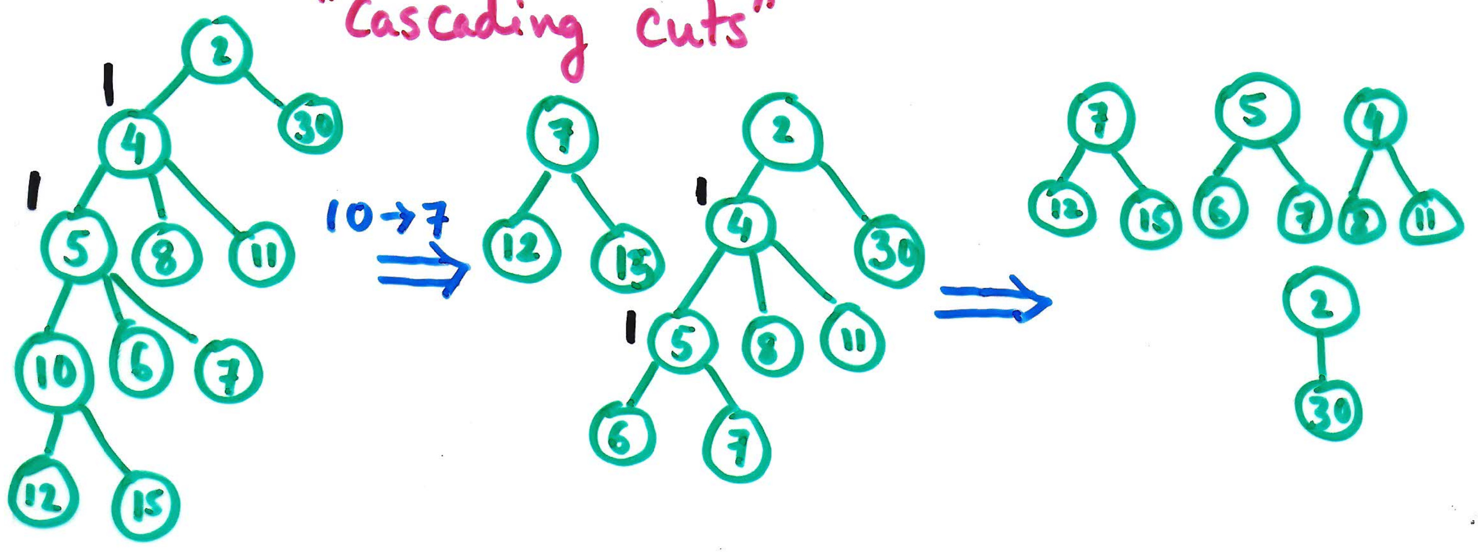
The problems with Decrease-key and Delete operations are that they destroy the binomial-heap property.

Cascading cuts. Let x be a node that becomes a child of another node at time t.

1. The first time x looses a child after t is marked

2. The second time x looses a child after t, we unlink x from its parent and add it to the root cycle.

Note: operation 2 may cause another operation 2 and hence "Cascading cuts"



# Bounding the maximum degree.

**Lemma 2.** Let $x$ be any node and $degree(x) = k$. Let $y_1, y_2, \dots y_k$ denote the children of $x$ in the order in which they were linked to $x$, from the earliest to the latest. Then,

$$degree(y_1) \geq 0 \text{ and } degree(y_i) \geq i-2$$

for $i = 2, 3, \dots, k$.

**Proof.** Before $y_i$ became a child of $x$, $x$ had had at least $i-1$ children and so had $y_i$. Later, $y_i$ lost at most one child and therefore its degree is at least $i-2$.

**Lemma 3.** Let $x$ be any node with $degree(x) = k$. Then, no. of descendants of $x$ including itself is at least $F_{k+2} \geq \phi^k$ where $\phi = \frac{1+\sqrt{5}}{2}$.

**Proof.** $s_k$ = # of descendants if degree $k$

$s_0 = 1, \quad s_1 = 2$

$s_k \geq 2 + \sum_{i=2}^{k} s_{i-2}$

By induction on $k$

$s_k \geq F_{k+2}$

Induction

Bases $k=0, k=1$

$$\begin{aligned} s_k &\geq 2 + \sum_{i=2}^{k} s_{i-2} \\ &\geq 2 + \sum_{i=2}^{k} F_i \\ &= 1 + \sum_{i=1}^{k} F_i \\ &= F_{k+2} \end{aligned}$$

Induction

$F_0 = 0$

$F_1 = 1$

$F_k = F_{k-1} + F_{k-2}$

**Corollary.** The maximum degree $D(n)$ of any node in an $n$-node F-heap is $O(\log n)$.

**Proof.** Let $x$ be any node with $\text{degree}(x) = k$.
We have $n \geq \#\text{descendants of } x \geq \phi^k$

$$\text{Thus, } k \leq \lfloor \log_\phi n \rfloor$$

$$D(n) = O(\log n)$$

## Time analysis of Decrease-key, delete

Actual cost $= O(c)$    $c$ calls of (cascading) cuts.

Increase in Pot. $= c$    more roots

Decrease in Pot. $= c-1$    nodes unmarked
(last cut might have marked a node)

$$\text{Change in Pot.} = c - 2(c-1)$$
$$= -c + 2$$

~~Actual~~ Amortized cost $= O(c) - c + 2 = O(1)$
with scaling