Fibonacci Heaps

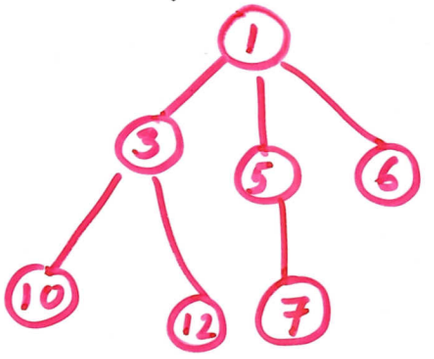
It supports the following operations efficiently in amortized sense.

1. Operations	Amortized time
Make-Heap	O(1)
Insert (H, x)	O(1)
Min (H)	O(1)
Meld (H_1, H_2)	O(1)
Delete-min(H)	O(log n)
Delete (H, x)	O(log n)
Decrease-key(H, x, Δ)	O(1)

2. Structure.

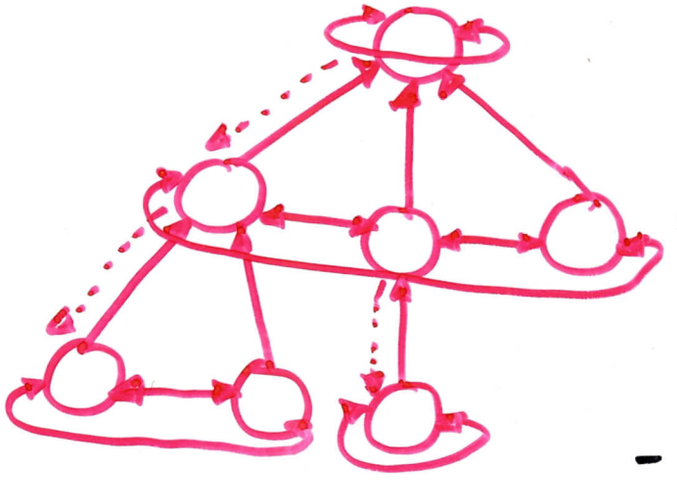
It is a collection of heap ordered trees.

Ex. (heap ordered tree)



parent has a lower key than its children

representation



- Siblings are doubly-linked in a cycle
- parent pointer
- Child pointer (one child)

The roots of the heap-ordered trees are linked in a doubly linked cycle plus a pointer to the minimum key node.

A single node x has:

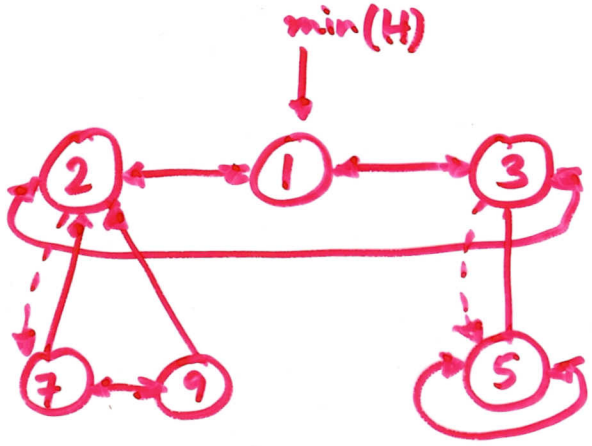
4 pointers (parent, child, l-sibling, r-sibling)
p(x) child(x) left(x) right(x)

1 real (key)
k(x)

1 integer (item)
i(x)

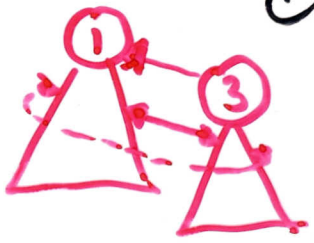
1 bit (marking)
mark(x)

1 integer (degree)
degree(x)



Simple Manipulations

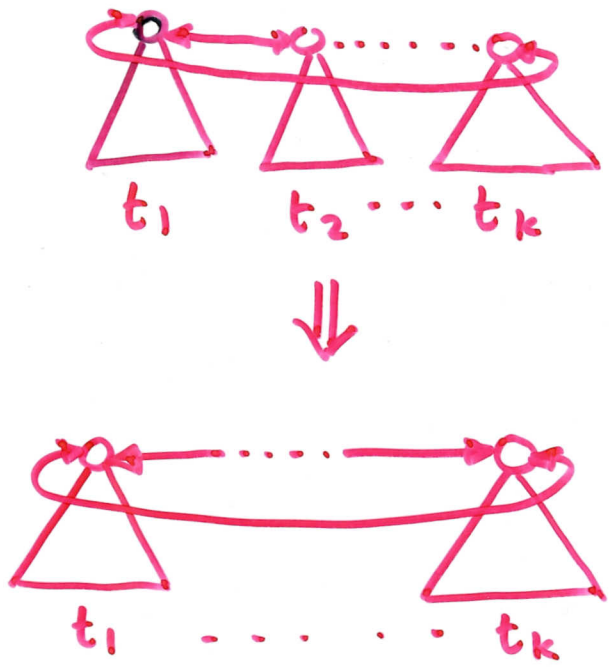
Linking: Given two heap-ordered trees, make root with bigger key the child of the other root.



O(1) time

Simple Manipulations.

Unlinking.



$O(1)$ time.

Merge Cycles. Cut open both cycles and connect at their ends.

Potential function. Let $z_1, z_2, \ldots z_n$ be the sequence of operations.

$t_i(H)$ = number of roots in the root list of H after operation z_i.

$m_i(H)$ = number of marked nodes in H after operation z_i.

P_i = potential
$= t_i(H) + 2m_i(H)$

Initial potential $P_0 = 0$ (Empty F-heap)

$T(z_i)$ = actual time for operation z_i

$A(z_i)$ = amortized time
$= T(z_i) + (P_i - P_{i-1})$

$$\sum_{i=1}^{n} T(z_i) = \sum_{i=1}^{n} (A(z_i) + (P_{i-1} - P_i))$$

$$= P_0 - P_n + \sum_{i=1}^{n} A(z_i)$$

$$= (t_0 - t_n) + 2(m_0 - m_n) + \sum_{i=1}^{n} A(z_i)$$

$$= -(t_n + 2m_n) + \sum_{i=1}^{n} A(z_i)$$

$$\sum_{i=1}^{n} T(z_i) \leq \sum_{i=1}^{n} A(z_i)$$

Operations

Make-heap: 1. Create null ptrs.
$P_0 = 0$
amortized cost = actual cost
$= O(1)$

Min(H): 1. return the minimum key in H
$P_i - P_{i-1} = 0$
amortized cost = actual cost
$= O(1)$

Meld(H_1, H_2): 1. Merge root cycles of H_1 & H_2
2. Adjust min(H) to be min(min(H_1), min(H_2))

Potential change $P_i - P_{i-1}$ =

$$(t_i + 2m_i) - (t_{i-1}(H_1) + 2m_{i-1}(H_1)) - (t_{i-1}(H_2) + 2m_{i-1}(H_2))$$

$$= 0$$

Amortized cost = Actual cost = $O(1)$

Insert(H, x) $\Rightarrow H'$

1. Create F-heap H_1 which has only one node x
2. Meld(H, H_1)

Let H' be the new F-heap

$$P_i(H') = t_{i-1}(H) + 1 + 2m_{i-1}(H)$$

$$P_i - P_{i-1} = 1$$

Amortized cost = 1 + actual cost = $O(1)$

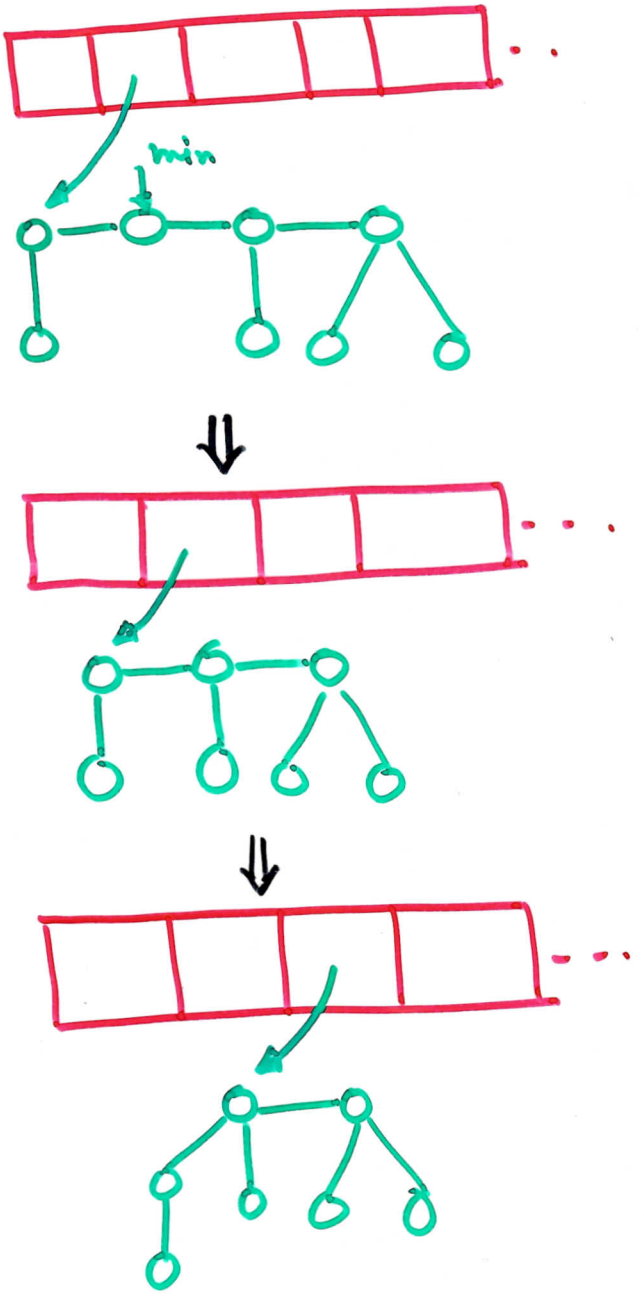
Delete-min(H)

1. remove the node with min. key from the root cycle
2. Merge the root cycle with the cycle of children of this node

⇓
continued..

3. While two roots r_1, r_2 have same degree
   link r_1, r_2
   endwhile

4. Adjust the minimum key



Step3. To find the roots with equal degrees, we use an array R; we scan the root list and let R[i] point to the root r with degree i. If R[i] is already occupied with root r', link r' with r and try to place it in R[i+1].

Cost analysis. Let $D(n)$ be the max. degree of any node in an n-node F-heap.

Actual cost

Step 1. $O(1)$
Step 2. $O(1)$
Step 3. $O(t_{i-1}(H) + D(n))$
Step 4. $O(t_{i-1}(H) + D(n))$

$\Big\}\ O(t_{i-1}(H) + D(n))$

Potential change

potential afterward is at most

$$D(n) + 1 + 2m_{i-1}(H)$$

potential before

$$t_{i-1}(H) + 2m_{i-1}(H)$$

potential change

$$D(n) + 1 - t_{i-1}(H)$$

Amortized cost: $O(t_{i-1}(H) + D(n)) + (D(n)+1) - t_{i-1}(H)$

$$= O(D(n)) + O(t_{i-1}(H)) - t_{i-1}(H)$$
$$= O(D(n))$$

since we can scale up the units of potential to dominate the constant hidden in $O(t_{i-1}(H))$.

$D(N) = O(\log n)$ for heaps used in F-heap.

Decrease key and Delete These two oprn. destroy the heap-ordered properties of trees in F-heap. However, the change is not too much.

Decrease-key(H, x, Δ).

1. Unlink tree rooted at x
2. Decrease the key by Δ
3. Add x to the root-cycle
4. Do cascading cuts. (not described) (Marking is used here).

Delete(H, x)

1. Unlink tree rooted at x
2. Add the cycle of children of x to the root cycle
3. dispose off x.

We skip the proof that these two oprn. take $O(\log n)$ amortized

Ex.