Heapsort

Based off slides by: David Matuszek

http://www.cis.upenn.edu/~matuszek/cit594-2008/

Presented by: Matt Boggus

Previous sorting algorithms

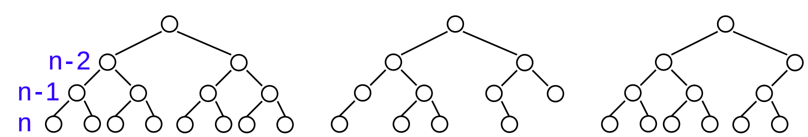
- Insertion Sort
  - $O(n^2)$ time
- Merge Sort
  - O(n) space

Heap data structure

- Binary tree
- Balanced
- Left-justified
- (Max) Heap property: no node has a value greater than the value in its parent

Balanced binary trees

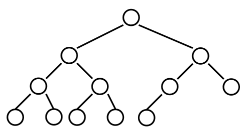
- Recall:
  - The depth of a node is its distance from the root
  - The depth of a tree is the depth of the deepest node
- A binary tree of depth `n` is balanced if all the nodes at depths `0` through `n-2` have two children



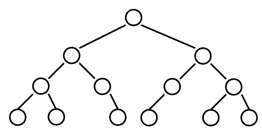
Balanced　　Balanced　　Not balanced

Left-justified binary trees

- A balanced binary tree of depth n is left-justified if:
  - it has 2^n nodes at depth n (the tree is “full”), or
  - it has 2^k nodes at depth k, for all $k < n$, *and* all the leaves at depth n are as far left as possible



Left-justified



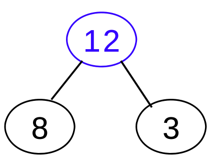
Not left-justified

Building up to heap sort

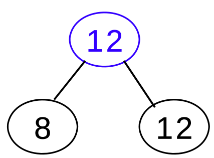
- How to build a heap
- How to maintain a heap
- How to use a heap to sort data

The heap property

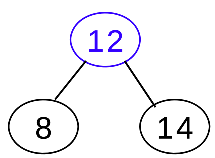
- A node has the heap property if the value in the node is as large as or larger than the values in its children



Blue node has heap property



Blue node has heap property

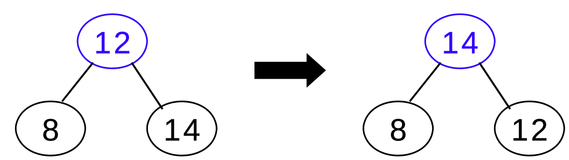


Blue node does not have heap property

- All leaf nodes automatically have the heap property
- A binary tree is a heap if *all* nodes in it have the heap property

siftUp

- Given a node that does not have the heap property, you can give it the heap property by exchanging its value with the value of the larger child



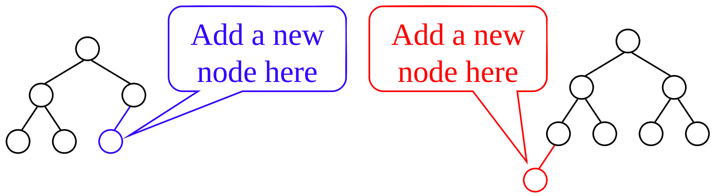
Blue node does not have heap property

Blue node has heap property

- This is sometimes called sifting up

Constructing a heap I

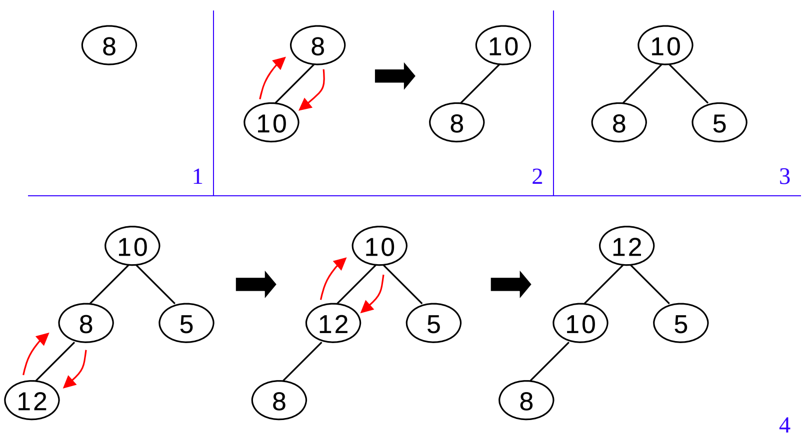
- A tree consisting of a single node is automatically a heap
- We construct a heap by adding nodes one at a time:
  - Add the node just to the right of the rightmost node in the deepest level
  - If the deepest level is full, start a new level
- Examples:



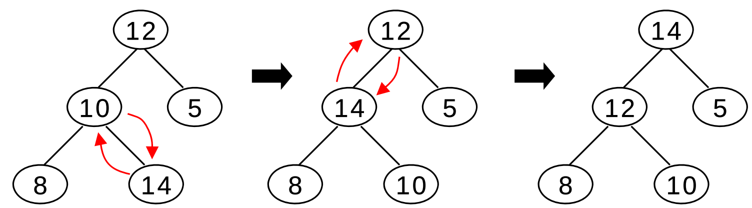
Constructing a heap II

- Each time we add a node, we may destroy the heap property of its parent node
- To fix this, we sift up
- But each time we sift up, the value of the topmost node in the sift may increase, and this may destroy the heap property of *its* parent node
- We repeat the sifting up process, moving up in the tree, until either
  - We reach nodes whose values don't need to be swapped (because the parent is *still* larger than both children), or
  - We reach the root

Constructing a heap III



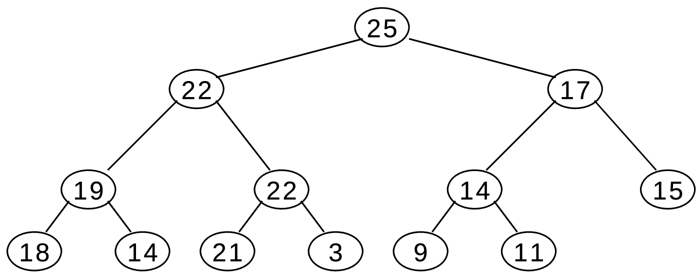
Other children are not affected



- The node containing 8 is not affected because its parent gets larger, not smaller
- The node containing 5 is not affected because its parent gets larger, not smaller
- The node containing 8 is still not affected because, although its parent got smaller, its parent is still greater than it was originally

A sample heap

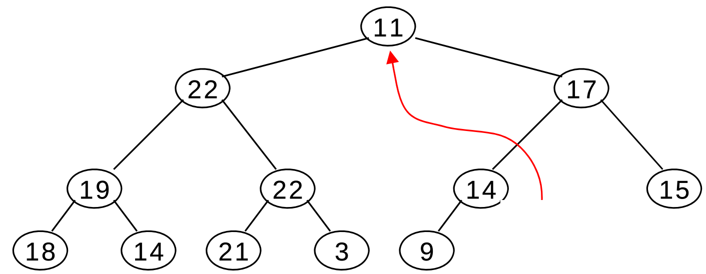
- Here's a sample binary tree after it has been heapified



- Notice that heapified does *not* mean sorted
- Heapifying does *not* change the shape of the binary tree; this binary tree is balanced and left-justified because it started out that way

Removing the root (animated)

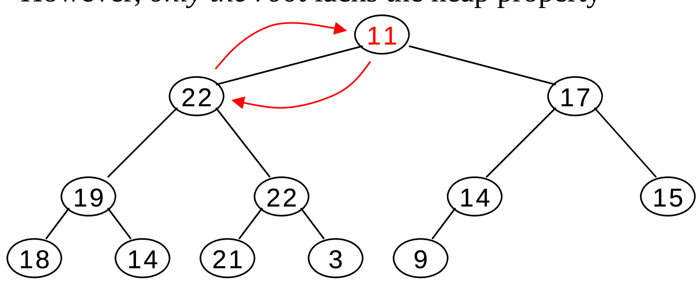
- Notice that the largest number is now in the root
- Suppose we *discard* the root:



- How can we fix the binary tree so it is once again *balanced and left-justified?*
- Solution: remove the rightmost leaf at the deepest level and use it for the new root

The `reHeap` method I

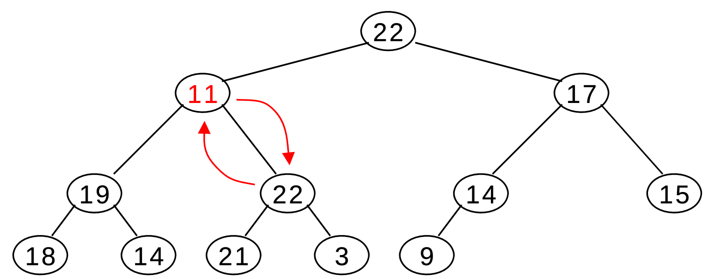
- Our tree is balanced and left-justified, but no longer a heap
- However, *only the root* lacks the heap property



- We can `siftUp()` the root
- After doing this, one and only one of its children may have lost the heap property

The `reHeap` method II

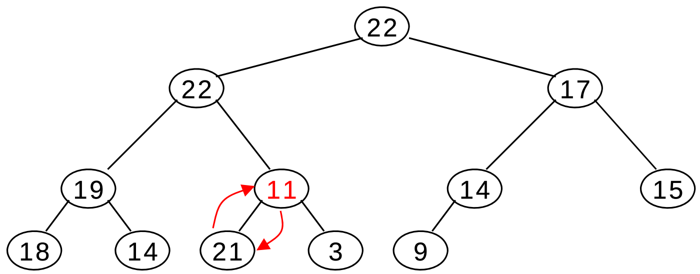
- Now the left child of the root (still the number `11`) lacks the heap property



- We can `siftUp()` this node
- After doing this, one and only one of its children may have lost the heap property

The reHeap method III

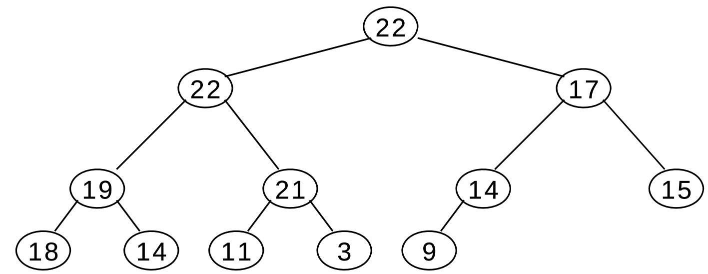
- Now the right child of the left child of the root (still the number 11) lacks the heap property:



- We can siftUp() this node
- After doing this, one and only one of its children may have lost the heap property —but it doesn't, because it's a leaf

The reHeap method IV

- Our tree is once again a heap, because every node in it has the heap property



- Once again, the largest (or *a* largest) value is in the root
- We can repeat this process until the tree becomes empty
- This produces a sequence of values in order largest to smallest

Sorting

- What do heaps have to do with sorting an array?
- Here's the neat part:
  - Because the binary tree is *balanced* and *left justified*, it can be represented as an array
    - ***Danger Will Robinson:*** This representation works well ***only*** with ***balanced***, ***left-justified*** binary trees
  - All our operations on binary trees can be represented as operations on *arrays*
  - To sort:

```
heapify the array;
while the array isn't empty {
   remove and replace the root;
   reheap the new root node;
}
```


Key properties

- Determining location of root and "last node" take constant time

- Remove n elements, re-heap each time

Analysis

- To reheap the root node, we have to follow *one path* from the root to a leaf node (and we might stop before we reach a leaf)
- The binary tree is perfectly balanced
- Therefore, this path is O(log n) long
  - And we only do O(1) operations at each node
  - Therefore, reheaping takes O(log n) times
- Since we reheap inside a while loop that we do n times, the total time for the while loop is n*O(log n), or O(n log n)

Analysis

- Construct the heap O(n log n)
- Remove and re-heap O(n log n)
- Total time O(n log n) + O(n log n)

The End

- Continue to priority queues?

Priority Queue

- Queue – only access element in front
- Queue elements sorted by order of importance
- Implement as a heap where nodes store priority values

Extract Max

- Remove root
- Swap with last node
- Re-heapify

Increase Key

- Change node value
- Re-heapify

Insert

- Add new node, priority is minimum possible value
- Increase priority

The End