Introduction to Algorithms
Solving Recursions

CSE 680

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Material adapted from Prof. Shafi Goldwasser, MIT

Substitution method

- *The most general method:*
  1. ***Guess*** the form of the solution.
  2. ***Verify*** (or refine) by induction.
  3. ***Solve*** for constants.
- ***Example:*** $T(n) = 4T(n/2) + 100n$
  - [Assume that $T(1) = \Theta(1)$.]
  - Guess $O(n^3)$. (Prove O and Ω separately.)
  - Assume that $T(k) \le ck^3$ for $k < n$.
  - Prove $T(n) \le cn^3$ by induction.

Example of substitution

$$
\begin{aligned}
T(n) &= 4T(n/2) + 100n \\
&\le 4c(n/2)^3 + 100n \\
&= (c/2)n^3 + 100n \\
&= cn^3 - ((c/2)n^3 - 100n) \quad \leftarrow \text{desired – residual} \\
&\le cn^3 \quad \leftarrow \text{desired}
\end{aligned}
$$

whenever $(c/2)n^3 - 100n \ge 0$, for example, if $c \ge 200$ and $n \ge 1$.

↖ residual

Example (continued)

- We must also handle the initial conditions, that is, ground the induction with base cases.
- ***Base:*** $T(n) = \Theta(1)$ for all $n < n_0$, where n_0 is a suitable constant.
- For $1 \le n < n_0$, we have "$\Theta(1)$" $\le cn^3$, if we pick c big enough.

This bound is not tight!

A tighter upper bound?

We shall prove that $T(n) = O(n^2)$.

Assume that $T(k) \le ck^2$ for $k < n$:

$$T(n) = 4T(n/2) + 100n$$
$$\le cn^2 + 100n$$
$$\le cn^2$$

for ***no*** choice of $c > 0$. Lose!

A tighter upper bound!

IDEA: Strengthen the inductive hypothesis.

- ***Subtract*** a low-order term.

Inductive hypothesis: $T(k) \le c_1k^2 - c_2k$ for $k < n$.

$$T(n) = 4T(n/2) + 100n$$
$$\le 4(c_1(n/2)^2 - c_2(n/2)) + 100n$$
$$= c_1n^2 - 2c_2n + 100n$$
$$= c_1n^2 - c_2n - (c_2n - 100n)$$
$$\le c_1n^2 - c_2n \quad \text{if } c_2 > 100.$$

Pick c_1 big enough to handle the initial conditions.

Recursion-tree method

- A recursion tree models the costs (time) of a recursive execution of an algorithm.
- The recursion tree method is good for generating guesses for the substitution method.
- The recursion-tree method can be unreliable, just like any method that uses ellipses (…).
- The recursion-tree method promotes intuition, however.

Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:

Example of recursion tree

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$T(n)$

Example of recursion tree

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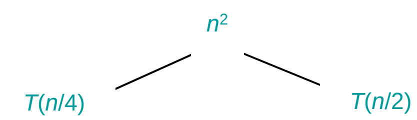

Example of recursion tree

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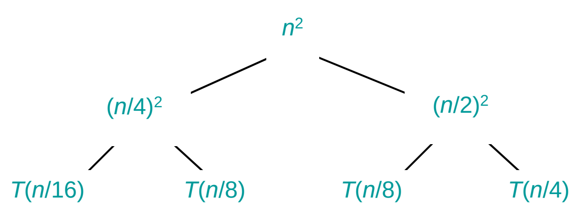

Example of recursion tree

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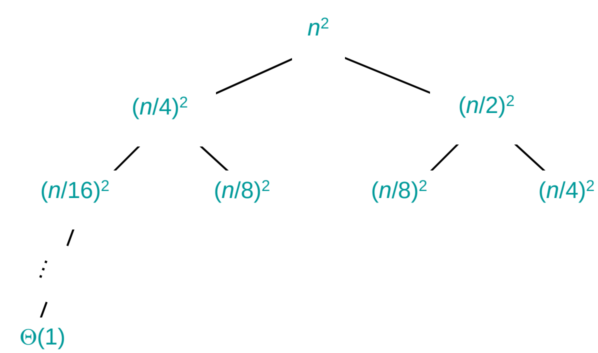

Example of recursion tree

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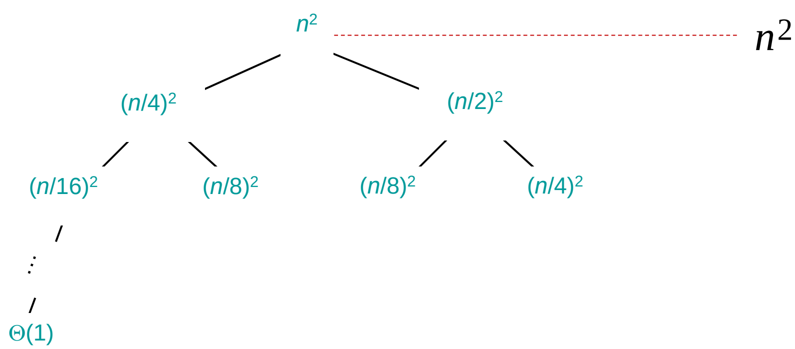

Example of recursion tree

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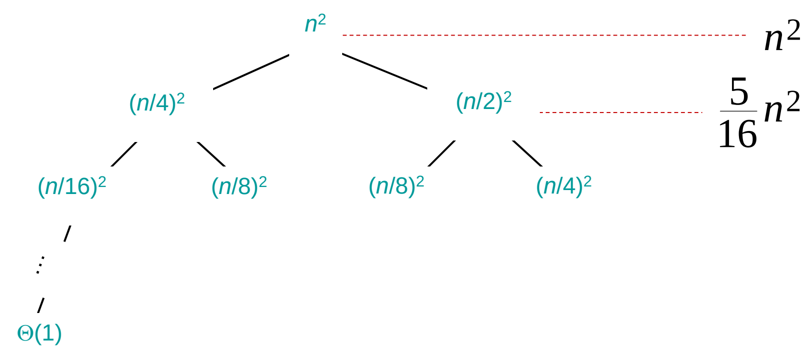
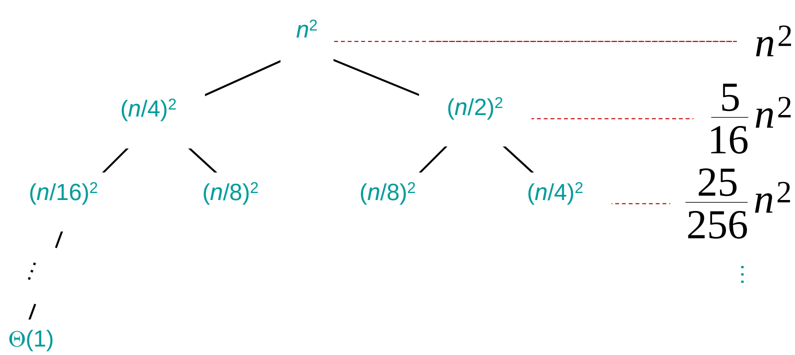


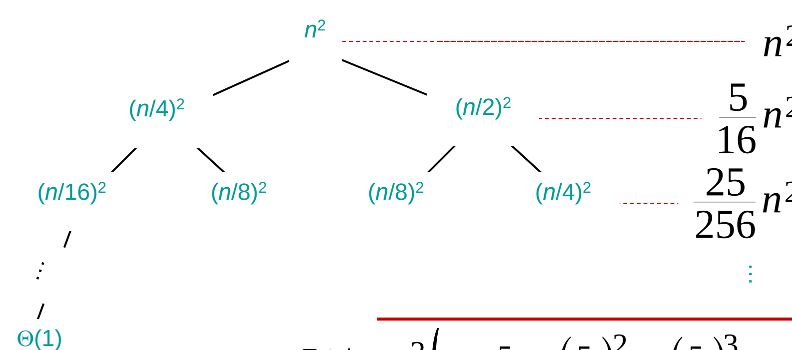
Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Example of recursion tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$:



Total $= n^2\left(1 + \frac{5}{16} + \left(\frac{5}{16}\right)^2 + \left(\frac{5}{16}\right)^3 + \cdots\right)$

$= \Theta(n^2)$ *geometric series*

Appendix: geometric series

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x} \quad \text{for } x \neq 1$$

$$1 + x + x^2 + \cdots = \frac{1}{1 - x} \quad \text{for } |x| < 1$$

The master method

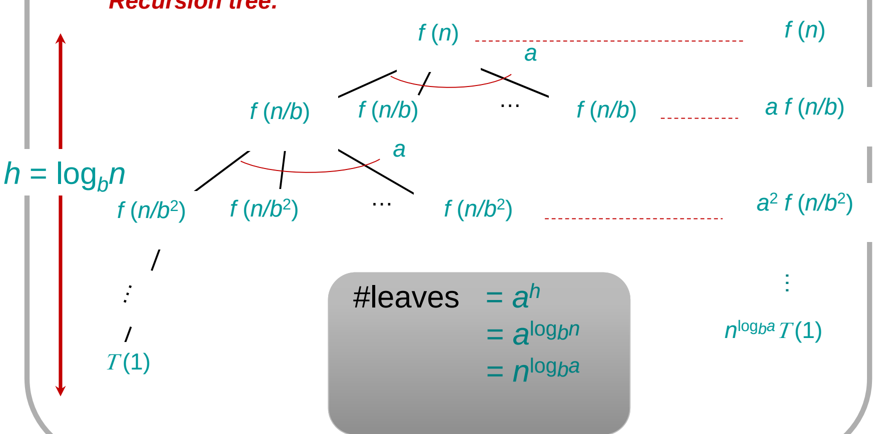
The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)\,,$$

where $a \geq 1$, $b > 1$, and f is asymptotically positive.

Idea of master theorem

Recursion tree:



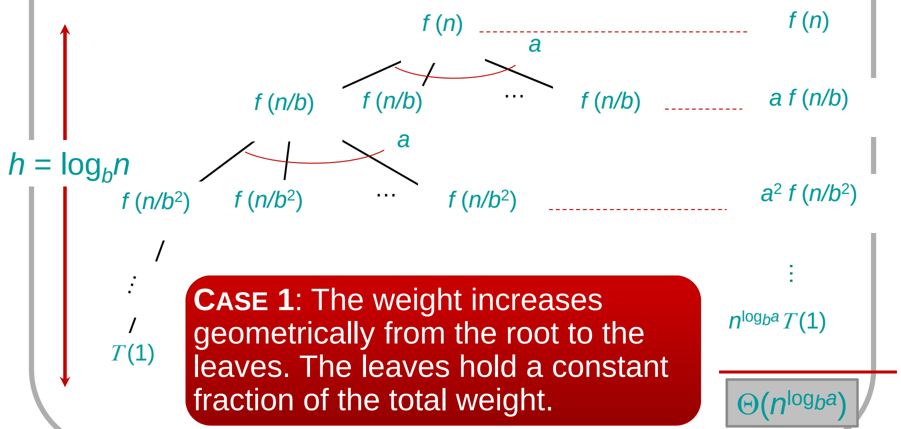
Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   Solution: $T(n) = \Theta(n^{\log_b a})$.

Idea of master theorem



CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

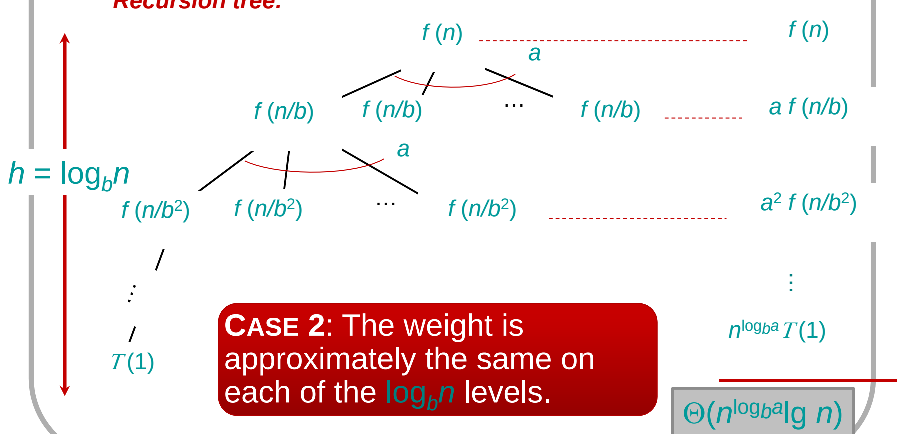
Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

2. If $f(n) = \Theta(n^{\log_b a})$
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

   Solution: $T(n) = \Theta(n^{\log_b a} \lg n)$.

Idea of master theorem



CASE 2: The weight is approximately the same on each of the $\log_b n$ levels.

Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

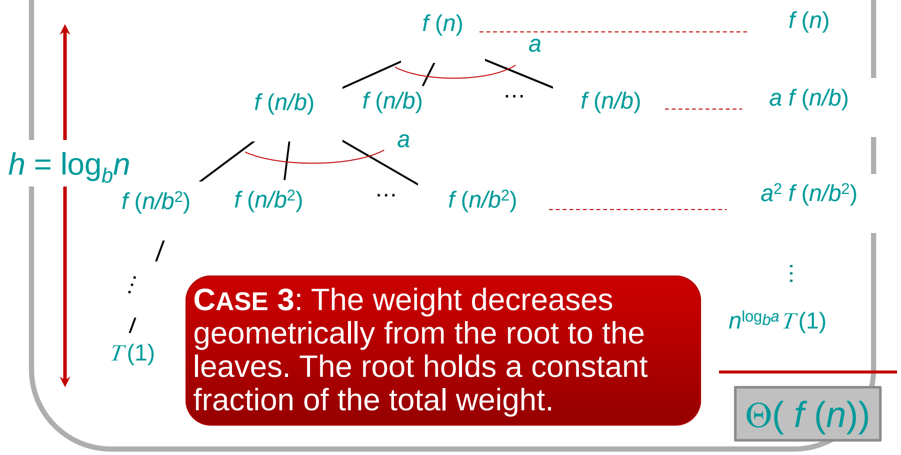
3. $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor),

   and $f(n)$ satisfies the ***regularity condition*** that $a f(n/b) \le c f(n)$ for some constant $c < 1$.

   Solution: $T(n) = \Theta(f(n))$.

Idea of master theorem

Recursion tree:



CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

Examples

Ex. $T(n) = 4T(n/2) + n$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n$.

$f(n) = O(n^{2-\varepsilon})$ for $\varepsilon = 1$ => Case 1

$\therefore T(n) = \Theta(n^2)$.

Ex. $T(n) = 4T(n/2) + n^2$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^2$.

$f(n) = \Theta(n^2)$ => Case 2

$\therefore T(n) = \Theta(n^2 \lg n)$.

Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^3$.

$f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$ => Case 3

and $4(cn/2)^3 \le cn^3$ (reg. cond.) for $c = 1/2$.

$\therefore T(n) = \Theta(n^3)$.

Ex. $T(n) = 4T(n/2) + n^2/\lg n$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^2/\lg n$.

Master method does not apply. In particular, for every constant $\varepsilon > 0$, we have $n^{\varepsilon} = \omega(\lg n)$.